International Journal of Theoretical Physics, Vol. 26, No. 4, 1987

Cosmological Transition Periods

Ronald Gautreau[1]

Received March 6, 1986

A method is given for continuously following a model of a universe that in its evolution makes a transition from one type of universe to another. As an illustration, a universe is considered that initially is radiation-dominated and then makes a transition to a final matter-dominated Einstein–de Sitter universe. The epoch when the universe changes from being radiation-dominated to being matter-dominated is found and is related to the epoch when radiation decouples from matter.

In describing how the universe evolves after an assumed Big Bang, the evolution is usually broken into different intervals, such as a "radiation-dominated" interval or a "matter-dominated" interval. These different intervals are then analyzed separately, without much discussion of the metric and particle motions in the transition period connecting one interval to another. In this paper we develop a method which, besides providing a convenient and simple way for treating the different intervals individually, gives a formalism for dealing with the continuous transition from a radiation-dominated to a matter-dominated universe. In particular, we are able to determine the epoch corresponding to the change in the universe from being radiation-dominated to being matter-dominated, and to relate this to the epoch where radiation decouples from matter.

We will consider only zero-curvature universes. The metric for such universes is usually written in the Robertson–Walker isotropic form as

$$ds^2(r, \tau) = e^{2h(\tau)}(dr^2 + r^2\,d\Omega^2) - d\tau^2$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2 \tag{1}$$

In this paper, however, we will make use of the results of a different formulation of cosmological theory that we have recently developed

[1]Physics Department, New Jersey Institute of Technology, Newark, New Jersey 07102.

(Gautreau, 1984a), where the metric for the universe has the form

$$ds^2(R, \tau) = [dR - m(R/B)\, d\tau]^2 + R^2\, d\Omega^2 - d\tau^2 \tag{2}$$

with

$$B(\tau) = (8\pi\rho/3 + \Lambda/3)^{-1/2} \tag{3}$$

where ρ is the density of the cosmological geodesic fluid that is the source of the particular universe under consideration, $m = +1$ (-1) for an expanding (collapsing) fluid, and Λ is the cosmological constant. The transformation that takes one from the (R, τ) coordinates of (2) to the (r, τ) coordinates of (1) is

$$R = re^{h(\tau)} \tag{4}$$

To be consistent with a Big Bang picture, from now on we will consider only expanding universes $(m = +1)$. Extension to a collapsing universe $(m = -1)$ is straightforward. The equation for the trajectory $R_g(\tau)$ of a radially moving geodesic particle of the expanding cosmological fluid is obtained by setting $ds^2(R, \tau) = -d\tau^2$ in (2) to obtain

$$dR_g/d\tau = R_g/B \tag{5}$$

In (2), both coordinates R and τ have very natural physical interpretations. The time coordinate τ is measured by clocks fixed in the cosmological geodesic fluid. On a subspace $\tau = \text{const}$, the metric (2) becomes

$$ds^2(R, \tau = \text{const}) = dR^2 + R^2\, d\Omega^2 \tag{6}$$

which is seen to be flat. Thus, in addition to being the spatial coordinate that explicitly exhibits the flatness of $\tau = \text{const}$ subspaces, R has the physical significance that it is equal to the proper distance between τ-simultaneous events in the universe under consideration. We have previously pointed out this significance of R for a Schwarzschild field (Gautreau and Hoffmann, 1978; see also Ftaclas and Cohen, 1980). Our formalism also allows the treatment of cosmological problems not addressable with (1), such as the inhomogeneous problem of describing a Schwarzschild mass imbedded in a given universe, in which the metric approaches a Schwarzschild field close to the imbedded mass and goes over to the given universe far from the Schwarzschild mass (Gautreau, 1984b; see also Van den Bergh and Wils, 1984). In addition, we have used our formalism to incorporate Dirac's Large Numbers hypothesis into Einstein's standard theory of general relativity (Gautreau, 1985).

From the Einstein field equations with the metric (2), the pressure p in the cosmological fluid is related to the fluid density ρ by (Gautreau, 1984a)

$$p + \rho + (24\pi\rho + 3\Lambda)^{-1/2}\, d\rho/d\tau = \rho + p + \tfrac{1}{3}B\, d\rho/d\tau = 0 \tag{7}$$

Combining (5) with (7), we obtain

$$\frac{d}{d\tau}(\rho R_g^3) + p\frac{d}{d\tau}(R_g^3) = 0 \tag{8}$$

Equations (7) and (8), coupled with a relationship between p and ρ, are the fundamental equations for the quantities ρ, p, and R_g in the (R, τ) formalism of (2).

Equation (8) is a statement of conservation of energy. As a galaxy evolves along the trajectory $R_g(\tau)$, the first law of thermodynamics requires that the change in energy dE of the expanding system between $0 \leq R \leq R_g$, whose volume is $V = \frac{4}{3}\pi R_g^3$, must be equal to the work done by the pressure in the system:

$$dE = -p\,dV = -\tfrac{4}{3}p\pi\,d(R_g^3) \tag{9}$$

The energy in the system is related to the mass M in the system by

$$E = Mc^2 = \rho V c^2 = \rho\tfrac{4}{3}\pi R_g^3 c^2 \tag{10}$$

Combining (9) with (10), we get (8) with $c = 1$. From now on, we will take $\Lambda = 0$.

Let us illustrate how some familiar universes fit into this formalism. In a radiation-dominated universe, the radiation density ρ_r is much greater than the matter density ρ_m, $\rho_r \gg \rho_m$, so that $\rho = \rho_r + \rho_m \approx \rho_r$, and the pressure $p = \rho_r/3$. For this case, the above expressions yield

$$B = 2\tau \tag{11}$$

$$\rho_r = 3/(32\pi\tau^2) \tag{12}$$

$$ds^2(R, \tau) = [dR - (R/2\tau)\,d\tau]^2 + R^2\,d\Omega^2 - d\tau^2 \tag{13}$$

and, from (5), the equation for the variation of proper distance R_g with time τ of a geodesic particle of the cosmological fluid is

$$R_g = b\tau^{1/2} \tag{14}$$

where b is a constant related to the energy of the particle at the Big Bang at $\tau = 0$. Note that $\rho_r R_g^4 = \text{const}$.

In a matter-dominated universe with zero pressure, i.e., an Einstein-de Sitter (ES) universe, where $\rho \approx \rho_m \gg \rho_r$, we obtain

$$B = 3\tau \tag{15}$$

$$\rho = 1/(6\pi\tau^2) \tag{16}$$

$$ds^2(R, \tau) = [dR - (2R/3\tau)\,d\tau]^2 + R^2\,d\Omega^2 - d\tau^2 \tag{17}$$

and, from (5), the equation of a particle of the cosmological fluid is

$$R_g = b\tau^{2/3} \tag{18}$$

with the constant b related to the energy that the particle has at the Big Bang at $\tau=0$. Note that $\rho R_g^3=\text{const}$, which is a statement that the gravitating mass inside an expanding sphere of radius R_g that attracts a galaxy on the surface of the sphere stays constant as the galaxy expands after the Big Bang. We have discussed the ES universe extensively elsewhere (Gautreau, 1984a, 1985).

We now consider the combined problem of a universe whose density consists of both matter and radiation, so that

$$\rho=\rho_m+\rho_r \tag{19}$$

We will assume that the pressure is produced by the radiation, with negligible contribution from the matter, so that

$$p=\rho_r/3 \tag{20}$$

Substituting (19) and (20) into (8), we obtain

$$\frac{d}{d\tau}(\rho_r R_g^4)+R_g\frac{d}{d\tau}(\rho_m R_g^3)=0 \tag{21}$$

If we assume that there is no conversion of matter into radiation and vice versa, the amount of matter inside the volume bounded by an expanding galaxy will be constant, $\rho_m R_g^3=\text{const}$, so that separately

$$\frac{d}{d\tau}(\rho_m R_g^3)=0 \tag{22a}$$

$$\frac{d}{d\tau}(\rho_r R_g^4)=0 \tag{22b}$$

From (22a) and (22b) we find

$$F\equiv\rho_m/\rho_r\propto R_g \tag{23}$$

We then obtain from (7)

$$dF/d\tau=\tfrac{4}{3}a(1+F)^{1/2}/F, \qquad a=\text{const} \tag{24}$$

When (24) is integrated from $\tau=0$ and the initial condition is imposed that $F=\rho_m/\rho_r=0$ at $\tau=0$ to be consistent with radiation domination in the early universe, one obtains a cubic equation for $F(\tau)$:

$$F^3-3F^2-4a\tau(a\tau-2)=0 \tag{25}$$

If $a\tau>2$, (25) has one real root for F, while if $a\tau<2$, there are three real roots. Choosing the root where $0\leq a\tau\leq 2$ that corresponds to $F(\tau)=0$ at

$\tau=0$ and $F(\tau)=3$ at $a\tau=2$, we obtain

$$F=\rho_m/\rho_r=1+2\cos(2\theta/3+4\pi/3)$$

$$\cos\theta=1-a\tau, \qquad 0\leq a\tau\leq 2 \tag{26a}$$

$$F=\rho_m/\rho_r=1+\{a\tau-1+[(a\tau-1)^2-1]^{1/2}\}^{2/3}$$
$$+\{a\tau-1-[(a\tau-1)^2-1]^{1/2}\}^{2/3}, \qquad a\tau\geq 2 \tag{26b}$$

Combining (22a) and (22b) with (7), we obtain

$$B=F/(dF/d\tau) \tag{27}$$

so that $B(\tau)$ is determined from (26). From (3) (with $\Lambda=0$) we then obtain

$$\rho_r=\frac{3(dF/d\tau)^2}{8\pi(1+F)F^2} \tag{28}$$

$$\rho_m=\frac{3(dF/d\tau)^2}{8\pi(1+F)F} \tag{29}$$

Thus, (26)-(29) completely specify the metric (2) and the densities ρ_m and ρ_r as a function of cosmological time τ, up to an as-yet-undetermined constant a.

In the limit $a\tau\to 0$, corresponding to a radiation-dominated universe, we have

$$F=(8a\tau/3)^{1/2} \tag{30a}$$

$$B=2\tau \tag{30b}$$

$$\rho_m=(6a)^{1/2}/(16\pi\tau^{3/2}) \tag{30c}$$

$$\rho_r=3/(32\pi\tau^2) \tag{30d}$$

In the limit $a\tau\to\infty$, where there is a matter-dominated universe,

$$F=(2a\tau)^{2/3} \tag{31a}$$

$$B=\tfrac{3}{2}\tau \tag{31b}$$

$$\rho_m=1/(6\pi\tau^2) \tag{31c}$$

$$\rho_r=1/[6\pi(2a)^{2/3}\tau^{8/3}] \tag{31d}$$

Thus, as expected, we obtain radiation-dominated or matter-dominated universes in the appropriate limits.

We can express the constant a in terms of the present age of the universe τ_n and the presently measured temperature T_n of the background microwave radiation. From the Planck law, the radiation density of the blackbody radiation in units of mass/volume is

$$\rho_r=8\pi^5(kT)^4/15h^3c^5 \tag{32}$$

Substituting this into (31d), and inserting Newton's gravitational constant G into the expressions, we obtain

$$a = \frac{h^{9/2} c^{15/2}}{2[(8\pi^5/15)6\pi G]^{3/2}(kT_n)^6 \tau_n^4} \tag{33}$$

Using the values $T_n = 2.7$ K and $\tau_n = 10^{10}$ years $= 3.14 \times 10^{17}$ sec, we get

$$a = 3.89 \times 10^{-12}\ \text{sec}^{-1} = 1.22 \times 10^{-4}\ \text{year}^{-1} \tag{34}$$

If we define the transition time τ_c for the universe to change from being radiation-dominated to being matter-dominated as the time when the radiation density equals the matter density, i.e., when $F = 1$, we find from (26a) that

$$a\tau_c = 1 - \cos(\pi/4) \tag{35}$$

Using the value of a in (34), we get

$$\tau_c = 7.51 \times 10^{10}\ \text{sec} = 2390\ \text{years} \tag{36}$$

corresponding to a density from (28) or (29) of

$$\rho_{mc} = \rho_{rc} = 4.81 \times 10^{-14}\ \text{kg/m}^3 \tag{37}$$

and a radiation temperature from (32) of

$$T_c = 4.89 \times 10^4\ \text{K} \tag{38}$$

It is interesting to compare this value of τ_c with the approximate value τ_c' that would be obtained by assuming that the expressions (31) for $a\tau \to \infty$ for a matter-dominated universe held exactly from the present epoch $\tau = \tau_n$ back to $\tau = \tau_c$. For this situation we find from (31c) and (31d) that

$$\rho_m/\rho_r = (\rho_{mn}/\rho_{rn})(\tau/\tau_n)^{2/3} \tag{39}$$

where ρ_{mn} and ρ_{rn} are, respectively, the matter and radiation densities at the present epoch $\tau = \tau_n$. Setting $\rho_m = \rho_r$, we obtain τ_c' as

$$\tau_c' = (\rho_{rn}/\rho_{mn})^{3/2} \tau_n \tag{40}$$

With $T_n = 2.7$ K in (32)

$$\rho_{rn} = 4.46 \times 10^{-31}\ \text{kg/m}^3 \tag{41}$$

Using $\tau_n = 10^{10}$ years in (31c), we get

$$\rho_{mn} = 8.06 \times 10^{-27}\ \text{kg/m}^3 \tag{42}$$

giving

$$\tau_c' = 4120\ \text{years} \tag{43}$$

This is larger than the value $\tau_c = 2390$ years in (36), showing that the effects of increasing radiation density become important as τ decreases from τ_n to τ_c.

In the standard Big Bang picture, it is assumed that matter and radiation are locked together in thermal equilibrium at the same temperature until the universe cools to around $T_d = 3000$ K, at which time ionized hydrogen recombines with free electrons to form hydrogen atoms. When this occurs, the free electrons are no longer available to interact with and scatter the radiation, thermal equilibrium cannot be maintained, and the universe becomes nearly transparent to radiation. The time corresponding to $T_d = 3000$ K is referred to as the epoch of decoupling of radiation and matter. Before or after decoupling, though, the radiation is blackbody with energy density related to temperature by (32). Substituting $T_d = 3000$ K into (32), we obtain

$$\rho_{rd} = 6.80 \times 10^{-19}\ \text{kg/m}^3 \tag{44}$$

The corresponding epoch τ_d can be found from (28). However, the temperature $T_c = 4.89 \times 10^4$ K in (38) at the transition epoch $\tau_c = 2390$ years in (36) shows that decoupling will occur well into the matter-dominated universe. In this case, we can use (31d) to find τ_d as

$$\tau_d = 270{,}000\ \text{years} \tag{45}$$

The evolution with cosmological time τ of various quantities is summarized in Table I.

It has been proposed by some researchers in the context of "inflationary" universes that in the very early universe the predominant form should be a de Sitter universe.[2] A de Sitter universe corresponds to $\rho = \rho_0 = \text{const}$ in (3) and (7), with or without a cosmological constant Λ. In a de Sitter

Table I. The Evolution with Cosmological Time τ of Various Quantities Discussed in this Paper.[a]

$a\tau$	τ (years)	$F(\tau) = \rho_m/\rho_r$	ρ_r (kg/m^3)	T_r (K)
0	0	0	∞	∞
0.293	2,390	1	4.81×10^{-14}	48,900
1	8,200	2	3.01×10^{-15}	24,470
2	16,340	3	5.94×10^{-16}	16,310
33	270,000	16.3	6.80×10^{-19}	3,000
1.22×10^6	10^{10}	1.81×10^4	4.46×10^{-31}	2.7

[a]The numerical values have been obtained using $a = 3.89 \times 10^{-12}\ \text{sec}^{-1} = 1.22 \times 10^{-4}\ \text{year}^{-1}$, given in (34).

[2]See Guth (1984) for a review of inflationary scenarios.

universe one can have $\rho_0 = 0$ and $\Lambda \neq 0$, or $p = -\rho_0$ with $\Lambda = 0$, or some combination of the two. In any case, the equations governing a de Sitter interval with transitions to other types of universes can be worked out in our formalism.

It should be noted, though, that the de Sitter universe is intrinsically *static*, which perhaps is not what one might expect in a violently expanding very early universe. The static nature of the de Sitter universe is not apparent with the (r, τ) and (R, τ) coordinates used in (1) and (2). To exhibit the static form explicitly, it is necessary to change the time coordinate from the geodesic time τ to a curvature time T such that the metric becomes diagonal in (R, T) coordinates. Further, it is questionable whether the metric forms (1) or (2) can be achieved by physically sensible geodesic clock reference systems in a de Sitter universe (Gautreau, 1983). We have discussed this in detail elsewhere (Gautreau, 1983), and so will not develop this further here.

REFERENCES

Ftaclas, C., and Cohen, J. M. (1980). *Physical Review D*, **21**, 2103.

Gautreau, R. (1983). *Physical Review D*, **27**, 764.

Gautreau, R. (1984a). *Physical Review D*, **29**, 186.

Gautreau, R. (1984b). *Physical Review D*, **29**, 198.

Gautreau, R. (1985). *International Journal of Theoretical Physics*, **24**, 877.

Gautreau, R., and Hoffmann, B. (1978). *Physical Review D*, **17**, 2552.

Guth, A. H. (1984). In Eleventh Texas Symposium on Relativistic Astrophysics, D. S. Evans, ed., *Annals of the New York Academy of Sciences*, **422**.

Van den Bergh, N., and Wils, P. (1984). *Physical Review D*, **29**, 3002.